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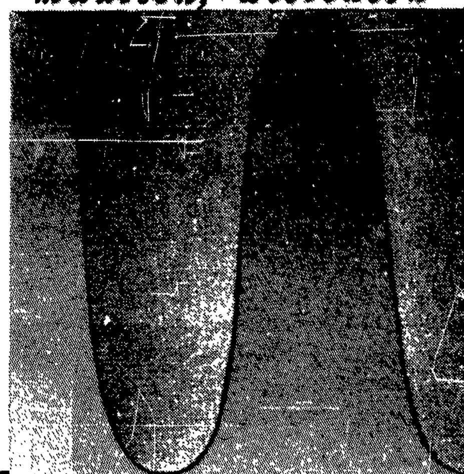

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DIVISION IN ALGEBRAS OF INFINITELY DIFFERENTIABLE FUNCTIONS

Walter Rudin

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I. Introduction

1.1 If $M_0, M_1, M_2, \ldots$ are positive numbers, we denote by $C\{M_n\}$ the class of all complex functions f on the real line for which there exist constants $\beta = \beta_f$ and $B = B_f$ such that

$$\|D^n f\| \leq \beta B^n M_n \qquad (n = 0, 1, 2, \ldots) \quad , \tag{1}$$

where $D = d/dx$ and $\| \quad \|$ is the supremum norm: $\|f\| = \sup|f(x)|$, $-\infty < x < \infty$.

The class of all members of $C\{M_n\}$ which are periodic, with period 2π, will be denoted by $C_p\{M_n\}$.

The sequence $\{M_n\}$ is said to be <u>logarithmically convex</u> if $\{\log M_n\}$ is convex, i.e., if $M_n^2 \leq M_{n-1}M_{n+1}$ for $n = 1, 2, 3, \ldots$. If $\{\bar{M}_n\}$ is the largest logarithmically convex minorant of $\{M_n\}$, then $C\{M_n\} = C\{\bar{M}_n\}$ and $C_p\{M_n\} = C_p\{\bar{M}_n\}$. This follows from the inequalities

$$\|D^n f\| \leq 2\|D^p f\|^{\frac{r-n}{r-p}} \|D^r f\|^{\frac{n-p}{r-p}} \qquad (0 \leq p \leq n < r) \tag{2}$$

which are due to Kolmogoroff [6; pp. 211, 216].

Hence we may assume, without loss of generality, that $\{M_n\}$ is logarithmically convex; unless the contrary is stated, this assumption will be made from now on.

Since $C\{M_n\} = C\{\lambda M_n\}$, for every positive constant λ, we may also assume

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that $M_0 = 1$. It will be convenient to define $A_0 = 1$ and

$$A_n = \left(\frac{M_n}{n!}\right)^{1/n} \qquad (n = 1, 2, 3, \ldots) \quad . \tag{3}$$

1.2. Leibnitz' formula

$$D^n(f \cdot g) = \sum_{j=0}^{n} \binom{n}{j} D^j f \cdot D^{n-j} g \tag{4}$$

shows that each $C\{M_n\}$ is an algebra, under pointwise addition and multiplication: the above assumptions on $\{M_n\}$ show that $M_j M_{n-j} \le M_n$ if $0 \le j \le n$, and therefore the inequalities $\|D^n f\| \le \beta_1 B_1^n M_n$ and $\|D^n g\| \le \beta_2 B_2^n M_n$ imply

$$\begin{aligned} \|D^n(f \cdot g)\| &\le \sum_{j=0}^{n} \binom{n}{j} \beta_1 B_1^j M_j \beta_2 B_2^{n-j} M_{n-j} \\ &\le \beta_1 \beta_2 M_n \sum_{j=0}^{n} \binom{n}{j} B_1^j B_2^{n-j} = \beta_1 \beta_2 (B_1 + B_2)^n M_n \quad . \end{aligned} \tag{5}$$

1.3. The algebra $C\{M_n\}$ is called <u>quasianalytic</u> if the zero-function is the only member of $C\{M_n\}$ such that $D^n f(x_0) = 0$ for $n = 0, 1, 2, \ldots$, at some point x_0. Otherwise , $C\{M_n\}$ is <u>non-quasianalytic</u>. The theorem of Denjoy and Carleman ([1], [6]) states that $C\{M_n\}$ is quasianalytic if and only if

$$\sum_{0}^{\infty} \frac{M_n}{M_{n+1}} = \infty \quad . \tag{6}$$

Since $\{\log M_n\}$ is convex and $M_0 = 1$, we see that $(M_n/M_{n+1})^n \le M_n^{-1}$, so that the condition (6) implies

$$\sum_{1}^{\infty} M_n^{-1/n} = \infty \quad . \tag{7}$$

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To prove the converse we appeal to the inequality [7]

$$\sum (a_1 a_2 \cdots a_n)^{1/n} \le e \sum a_n ,$$

valid for $a_i > 0$, and take $a_i = M_{i-1}/M_i$.

Thus (7) is also a necessary and sufficient condition for quasianalyticity.

1.4. If $1/f \in C\{M_n\}$ whenever $f \in C\{M_n\}$ and $\inf_x |f(x)| > 0$, we call $C\{M_n\}$ <u>inverse-closed</u>; a similar definition applies to $C_p\{M_n\}$.

The problem with which we are concerned, and which is solved in the present paper, is the description of all inverse-closed non-quasianalytic algebras $C\{M_n\}$. It turns out that they are precisely those for which there is a constant K such that the inequalities

$$A_s \le KA_n \tag{8}$$

hold whenever $s \le n$; here $\{A_n\}$ is defined by (3).

The condition (8) is satisfied with $K = 1$ precisely when $\{A_n\}$ is an increasing sequence. Accordingly, we shall call $\{A_n\}$ <u>almost increasing</u> if (8) is satisfied for some $K < \infty$.

1.5. Actually, a more striking dichotomy exist than was indicated in the preceding paragraph. Our main results may be summarized as follows:

THEOREM A. <u>Suppose</u> $\{A_n\}$ <u>is almost increasing</u>. <u>Then</u> $C\{M_n\}$ <u>is inverse-closed. Furthermore, if</u> $f \in C\{M_n\}$ <u>and if</u> ϕ <u>is an analytic function in an open set which contains the closure of the range of</u> f, <u>then</u> $\phi \circ f \in C\{M_n\}$.

THEOREM B. *Suppose* $C\{M_n\}$ *is non-quasianalytic and* $\{A_n\}$ *is not almost increasing. Then there exists an* $f \in C_p\{M_n\}$ *and an entire function* ϕ *such that*

(i) *if* λ *is any complex number, then* $(\lambda - f)^{-1}$ *is not in* $C\{M_n\}$;

(ii) $\phi \circ f$ *is not in* $C\{M_n\}$.

The symbol $\phi \circ f$ indicates the function defined by: $(\phi \circ f)(x) = \phi(f(x))$.

Since f is bounded, (i) shows that $C\{M_n\}$ is not inverse-closed, by taking $|\lambda| > \|f\|$. Actually, (i) shows more: for some $f \in C_p\{M_n\}$ the spectrum of f (relative to the algebra $C_p\{M_n\}$) consists of the whole plane, although the range of f is compact. We state the result for $C_p\{M_n\}$ rather than for $C\{M_n\}$ to emphasize that the phenomenon (i) is not caused by the behavior of f near infinity, but that it is present in non-quasianalytic algebras on the circle.

It would be interesting to extend Theorem B to quasianalytic classes.

1.6. The problem treated here has the following background. Let A be the class of all functions on the circle which are sums of absolutely convergent trigonometric series. Katznelson ([4], [2]) proved that if ϕ is defined on the real line and if $\phi \circ f \in A$ for all real $f \in A$, then ϕ must be analytic on the line. Malliavin [5] has proved that corresponding to every inverse-closed non-quasianalytic class $C\{M_n\}$ there is a real $f \in A$ such that $\phi \circ f \in A$ only if $\phi \in C\{M_n\}$. It is known that the intersection of all non-quasianalytic classes is precisely the class $C\{n!\}$, which consists of analytic functions (a proof is included in Part IV). If it were true that the intersection of all inverse-closed non-quasianalytic classes is also $C\{n!\}$, then Malliavin's result would imply Katznelson's. But it is not so:

THEOREM C. <u>The intersection of all inverse-closed non-quasianalytic classes is precisely the class</u> $C\{(n \log n)^n\}$.

Since $C\{M_n\}$ is a subclass of $C\{M_n^*\}$ if and only if $\{(M_n/M_n^*)^{1/n}\}$ is bounded above [1;p.19] and since Stirling's formula implies that

$$n \cdot \log n \cdot (n!)^{-\frac{1}{n}} \to \infty ,$$

we see that $C\{n!\}$ is a proper subclass of $C\{(n \log n)^n\}$.

In particular, it follows that <u>there exist non-quasianalytic algebras which are not inverse-closed</u>, a fact which seems to have escaped previous notice.

II. PROOF OF THEOREM A.

2.1. THEOREM. <u>Suppose</u> $A_s \le KA_n$ <u>whenever</u> $s \le n$, <u>for some fixed</u> K. <u>If</u> σ, β, B <u>are positive constants, if</u>

(1) $$\|D^n f\| \le \beta B^n M_n \quad (n = 0, 1, 2, \ldots)$$

<u>and if</u> $|f(x)| \ge \sigma$ $(-\infty < x < \infty)$, <u>then</u>

(2) $$\|D^n(1/f)\| \le \beta_1 B_1^n M_n \quad (n = 0, 1, 2, \ldots) ,$$

<u>where</u> $\beta_1 = 2/\sigma$, $B_1 = BK(1 + 2\beta/\sigma)$.

This is due to Malliavin [5]. We include the proof since the quantitative version stated here is needed for Theorem 2.3.

<u>Proof.</u> Choose ϵ so that $2\beta\epsilon = (1 - \epsilon)\sigma$, then choose $\{r_n\}$ so that $BKA_n r_n = \epsilon$ $(n = 0, 1, 2, \ldots)$. Fix n, fix x_0, and define

$$Q(z) = f(x_0) + Df(x_0)z + \ldots + \frac{D^n f(x_0)}{n!} z^n .$$

For $1 \leq s \leq n$ we have

$$|D^s f(x_0)|/s! \leq \beta B^s A_s^s \leq \beta(BKA_n)^s$$

and hence $|z| \leq r_n$ implies

$$|Q(z)| \geq \sigma - \beta \sum_{s=1}^{n} (BKA_n r_n)^s > \sigma - \beta \sum_{1}^{\infty} \epsilon^s$$

(3)

$$= \sigma - \frac{\beta\epsilon}{1-\epsilon} = \frac{\sigma}{2} \quad .$$

The first n derivatives of Q at $z = 0$ are equal to the first n derivatives of f at $x = x_0$. Hence $D^n(1/f)(x_0) = D^n(1/Q)(0)$, and Cauchy's formula gives

$$D^n(1/f)(x_0) = \frac{n!}{2\pi i} \int_{|z|=r_n} \frac{dz}{z^{n+1}Q(z)} \quad . \tag{4}$$

We conclude from (3) and (4) that

$$|D^n(1/f)(x_0)| \leq \frac{2}{\sigma} \cdot \frac{n!}{r_n^n} = \frac{2}{\sigma}\left(\frac{BK}{\epsilon}\right)^n M_n \ ,$$

which completes the proof.

2.2. LEMMA. Suppose $\{f_p\}$ is a sequence of functions on the real line which converges pointwise to a function f, and which satisfies the inequalities

$$\|D^n f_p\| \leq R_n < \infty \quad (n = 0,1,2,\ldots;\ p = 1,2,3,\ldots) \tag{5}$$

Then we also have $\|D^n f\| \leq R_n$ for all $n \geq 0$.

Proof. Suppose that $D^j f$ exists and that $D^j f_p \to D^j f$ pointwise (for $j = 0$, this is part of the hypothesis). Fix x and $\epsilon > 0$, restrict y so that $0 < |y - x| < \epsilon/R_{j+2}$. Then

$$\frac{D^j f_p(y) - D^j f_p(x)}{y - x} - D^{j+1} f_p(x) = \frac{y-x}{2} D^{j+2} f_p(\xi) \tag{6}$$

for some ξ between x and y. Write (6) once more, with q in place of p, and subtract the two equations. The right side is less than ϵ; letting $p, q \to \infty$, the quotients on the left converge to the same limit, namely $\{D^j f(y) - D^j f(x)\}/(y-x)$. Hence $\{D^{j+1} f_p(x)\}$ is a Cauchy sequence. Let L be its limit. Then (6) gives

$$\left| \frac{D^j f(y) - D^j f(x)}{y - x} - L \right| \leq \epsilon \tag{7}$$

as soon as $0 < |y - x| < \epsilon/R_{j+2}$. Thus $D^{j+1} f$ exists and $D^{j+1} f_p \to D^{j+1} f$ pointwise.

The proof is completed by induction.

2.3. THEOREM. Suppose $f \in C\{M_n\}$, $\{A_n\}$ is almost increasing, and ϕ is analytic in an open set which contains the closure of the range of f. Then $\phi \circ f \in C\{M_n\}$.

Proof. There exists Γ, a union of finitely many rectifiable curves in the domain of ϕ, and there exists $\sigma > 0$, such that

$$|z - f(x)| \geq \sigma \tag{8}$$

for all $z \in \Gamma$ and all real x, and such that

$$\phi(f(x)) = \frac{1}{2\pi i} \int_\Gamma \frac{\phi(z)}{z - f(x)}\, dz \quad (-\infty < x < \infty) . \tag{9}$$

There is a sequence of partitions of Γ, by points $z_0^{(p)}, z_1^{(p)}, \ldots, z_{N_p}^{(p)}$, such that the functions g_p defined by

$$g_p(x) = \frac{1}{2\pi i} \sum_{j=1}^{N_p} \frac{\phi(z_j)}{z_j - f(x)} \left(z_j^{(p)} - z_{j-1}^{(p)}\right) \tag{10}$$

converge to $\phi(f(x))$, as $p \to \infty$.

Choosing β and B so that $\|D^n f\| \leq \beta B^n M_n$ and $\|f - z\| \leq \beta$ for all $z \in \Gamma$, Theorem 2.1 shows that

$$\|D^n(\frac{1}{z_j - f})\| \leq \beta_1 B_1^n M_n \qquad (n \geq 0) \quad . \tag{11}$$

Since ϕ is bounded on Γ and since $\sum |z_j^{(p)} - z_{j-1}^{(p)}|$ does not exceed the length of Γ, we see from (10) and (11) that

$$\|D^n g_p\| \leq \beta_2 B_1^n M_n \qquad (n \geq 0) \quad . \tag{12}$$

Lemma 2.2 now implies that

$$\|D^n(\phi \circ f)\| \leq \beta_2 B_1^n M_n \qquad (n \geq 0) \quad , \tag{13}$$

and this completes the proof.

III. PROOF OF THEOREM B.

3.1. LEMMA. Suppose $\{\alpha(n)\}$ is a sequence of positive numbers such that $\{n\alpha(n)\}$ is increasing but $\{\alpha(n)\}$ is not almost increasing. Then there exist sequences of integers, $\{s_i\}$ and $\{m_i\}$, both tending to ∞, such that

$$\frac{\alpha(s_i)}{\alpha(m_i s_i)} \to \infty \qquad (i \to \infty) \quad . \tag{1}$$

Proof. Put

$$\gamma(s) = \sup\{\frac{\alpha(s)}{\alpha(s+1)}, \frac{\alpha(s)}{\alpha(s+2)}, \frac{\alpha(s)}{\alpha(s+3)}, \ldots\} \quad . \tag{2}$$

Since $\{\alpha(n)\}$ is not almost increasing, we have $\sup_s \gamma(s) = \infty$.

Since $\{n\alpha(n)\}$ increases, we have

(3) $$\frac{\alpha(s)}{\alpha(ms)} \le m \qquad (m \ge 1) \ .$$

Also, if $s \le n$, then $n = ms + t$ with $0 \le t < s$, and so $\alpha(ms) \le n\alpha(n)/ms \le 2\alpha(n)$. Thus $\alpha(s)/\alpha(ms) \ge \alpha(s)/2\alpha(n)$, which gives

(4) $$\sup_{m \ge 1} \frac{\alpha(s)}{\alpha(ms)} \ge \frac{1}{2}\gamma(s) \ .$$

Since $\sup \gamma(s) = \infty$, (4) shows that (1) holds for some sequences $\{s_i\}$, $\{m_i\}$; by (3) this is only possible if $m_i \to \infty$.

If $\gamma(s) = \infty$, for all s, we can take for $\{s_i\}$ any sequence tending to ∞, and then find $\{m_i\}$ so that (1) holds. If $\gamma(s_0) < \infty$ for some s_0, then $\inf \alpha(n) > 0$, and (1) implies that $\alpha(s_i) \to \infty$, i.e., that $s_i \to \infty$.

3.2 LEMMA. <u>Suppose</u> $C\{M_n\}$ <u>is non-quasianalytic and</u> I <u>is a closed interval in the interior of a closed interval</u> J <u>on the real line. Then there exists a constant</u> β <u>and a function</u> h <u>such that</u> $h(x) = 1$ <u>on</u> I, $h(x) = 0$ <u>off</u> J, $0 \le h \le 1$, <u>and</u>

(5) $$\|D^n h\| \le \beta M_n \qquad (n = 0, 1, 2, \ldots) \ .$$

<u>Proof.</u> Put $a_n = M_{n-1}/M_n$. Then $\{a_n\}$ decreases monotonically, and $\sum a_n < \infty$. There exists a monotonically decreasing sequence $\{b_n\}$ such that $a_n/b_n \to 0$ and $\sum b_n < \infty$. Put $M_n^* = (b_1 b_2 \ldots b_n)^{-1}$. Then $\sum M_{n-1}^*/M_n^* = \sum b_n < \infty$ and $\{M_n^*\}$ is logarithmically convex. Hence $C\{M_n^*\}$ is non-quasianalytic. Also,

(6) $$\left\{\frac{M_n^*}{M_n}\right\}^{1/n} = \left\{\frac{a_1 \ldots a_n}{b_1 \ldots b_n}\right\}^{1/n} \to 0 \quad (n \to \infty) \ .$$

Since $C\{M_n\}$ is non-quasianalytic, there is a function $g \in C\{M_n^*\}$ such that $g(x) = 0$ if $x \leq 0$, $g(x) = 1$ if $x \geq x_0$ for some $x_0 > 0$. Bang [1; p.55] (see also Mandelbrojt [6; p.103]) has indicated a very simple construction which achieves this. Affine changes of variables (which do not affect the class $C\{M_n^*\}$) then give functions h_1, $h_2 \in C\{M_n^*\}$ such that $h_1 = 0$ to the left of J, $h_1 = 1$ on I and to the right of I, $h_2 = 0$ to the right of J, $h_2 = 1$ on I and to the left of I. Put $h = h_1 h_2$. Then h has the required properties, except that (5) is replaced by

$$\|D^n h\| \leq B^n M_n^* \qquad (n = 0, 1, 2, \ldots) \ , \tag{7}$$

for some constant B. Setting $\beta = \max_n B^n M_n^*/M_n$, (6) shows that $\beta < \infty$, and (7) shows that (5) holds.

3.3. We now turn to the proof of Theorem B. Put

$$\mu_n = M_n/M_{n+1} \qquad (n = 0, 1, 2, \ldots) \ . \tag{8}$$

By the Denjoy-Carleman Theorem, $\sum \mu_n < \infty$. Replacing M_n by $k^n M_n$, if necessary, we may assume, without loss of generality, that

$$\sum_0^\infty \mu_n < \frac{1}{2} \ . \tag{9}$$

We define

$$f_s(x) = \mu_s^s M_s \exp\{ix/\mu_s\} \qquad (s = 0, 1, 2, \ldots) \ , \tag{10}$$

and note that

$$D^n(f_s^m) = (im/\mu_s)^n f_s^m \qquad (s, n \geq 0, \quad m \geq 1) \ . \tag{11}$$

The convexity of $\{\log M_n\}$ shows that $M_s^{s+1-n} \leq M_n M_{s+1}^{s-n}$ if $0 \leq n \leq s$; if $s + 1 \leq n$, we have similarly $M_{s+1}^{n-s} \leq M_s^{n-s-1} M_n$. Thus the inequality

(12) $$M_s^{s+1-n} \leq M_n M_{s+1}^{s-n}$$

holds in all cases.

Applying (12) to (11), with $m = 1$, we see that

(13) $$\| D^n f_s \| = \mu_s^{s-n} M_s \leq M_n \qquad (s, n \geq 0) .$$

In particular, taking $n = 0$,

(14) $$\| f^m_s \| = \| f_s \| \leq M_0 = 1 \qquad (s \geq 0, \quad m \geq 1) .$$

By (9), we can place disjoint closed intervals J_k in $(0, 2\pi)$ which contain intervals I_k in their interiors, with $m(I_k) = 2\pi \mu_k$, and Lemma 3.2 shows that there are functions h_k and constants $\beta_k > k$ such that $h_k = 1$ on I_k, $h_k = 0$ off J_k, and

(15) $$\| D^n h_k \| \leq \beta_k M_n \qquad (u, k \geq 0) .$$

Put $\alpha(0) = 1$ and define $\alpha(n)$ by

(16) $$n \, \alpha \, (n) = M_n^{1/n} \qquad (n \geq 1) .$$

By hypothesis, $\{A_n\}$ is not almost increasing. By Stirling's formula, $\{\alpha(n)/A_n\}$ is bounded above and below by positive numbers. Hence $\{\alpha(n)\}$ is not almost increasing. Our standing assumptions on $\{M_n\}$ (logarithmic convexity, and $M_0 = 1$) imply that $\{n \, \alpha(n)\}$ increases. Thus Lemma 3.1 applies, and there are sequences $\{s_k\}$, $\{m_k\}$, tending to ∞, such that $s_k > k$, $2^{s_k} > \beta_k$, and

(17) $$\frac{\alpha(s_k)}{\alpha(m_k s_k)} \to \infty \qquad (k \to \infty) .$$

We extend the functions $h_k \cdot f_{s_k}$, defined in $(0, 2\pi)$, to be periodic, with period 2π, and define

$$(18) \qquad f(x) = \sum_{k=0}^{\infty} \frac{1}{\beta_k} h_k(x) f_{s_k}(x) \ .$$

By (13), (15), and Leibnitz' formula, we have $\| D^n(h_k f_{s_k}) \| \leq 2^n M_n$. The functions h_k have disjoint supports. Hence if g is any partial sum of the series (18), we have $\| D^n g \| \leq 2^n M_n$, and we conclude from Lemma 2.2 that $\| D^n f \| \leq 2^n M_n$. Thus $f \in C_p\{M_n\}$.

Since 0 is in the range of f, it is clear that f^{-1} is not in $C\{M_n\}$. Fix $\lambda \neq 0$, put $F = (1 - f/\lambda)^{-1}$, and assume (this will lead to a contradiction) that $F \in C\{M_n\}$. For some $B < \infty$ we then have

$$(19) \qquad \| D^n F \| \leq B^n M_n \qquad (n \geq 1) \ .$$

For large enough k, $|\lambda| \beta_k > 1$. Since $h_k = 1$ on I_k and $h_j = 0$ on I_k if $j \neq k$, we have

$$(20) \qquad F(x) = \sum_{m=0}^{\infty} (\lambda \beta_k)^{-m} f^m_{s_k}(x) \qquad (x \in I_k, \ \ k \geq k_0) \ .$$

By (11) and (14), the series (20) may be differentiated term by term any number of times, since the resulting series converge uniformly on I_k. Since $s_k > k$, we have $\mu_{s_k} \leq \mu_k$, so that there is a point $x_k \in I_k$ at which $\exp\{ix/\mu_{s_k}\} > 0$.

Differentiating (20) n times at x_k therefore gives

$$(21) \qquad D^n F(x_k) = i^n \sum_{m=0}^{\infty} (m/\mu_{s_k})^n | f_{s_k}(x_k)/\lambda\beta_k |^m \ ,$$

by (11). By (19), no term in the series (21) exceeds $B^n M_n$. Taking $m = m_k$ and $n = m_k s_k$, (10) shows therefore that

$$\left| \frac{m_k^{s_k} M_{s_k}}{\lambda \beta_k} \right|^{m_k} \leq B^{m_k s_k} M_{m_k s_k} \qquad (k \geq k_0) . \tag{22}$$

Taking n^{th} roots in (22) and using (16), we obtain

$$\frac{\alpha(s_k)}{\alpha(m_k s_k)} \leq B |\lambda \beta_k|^{1/s_k} \leq 2B|\lambda|^{1/s_k} . \tag{23}$$

The last term in (23) is bounded, as $k \to \infty$, and this contradicts (17).

Thus $(1 - f/\lambda)^{-1}$ is not in $C\{M_n\}$, and part (i) of Theorem B is proved.

Part (ii) is proved quite similarly. Suppose

$$\phi(z) = \sum_0^\infty c_m z^m , \quad 0 < c_m < 1, \quad c_m^{1/m} \to 0 , \tag{24}$$

and put $g(x) = \phi(f(x))$. On I_k we have, in place of (20),

$$g(x) = \sum_{m=0}^\infty \frac{c_m}{\beta_k^m} f_{s_k}^m (x) , \tag{20'}$$

and we can choose $x_k \in I_k$ so that $f_{s_k}(x_k) > 0$. In place of (23) we obtain

$$c_{m_k}^{1/m_k s_k} \cdot \frac{\alpha(s_k)}{\alpha(m_k s_k)} \leq 2B . \tag{23'}$$

Since $c_m^{1/m} \leq c_m^{1/ms}$, this gives

$$c_{m_k}^{1/m_k} \leq 2B \cdot \frac{\alpha(m_k s_k)}{\alpha(s_k)} . \tag{25}$$

But $\{c_m^{1/m}\}$ can tend to 0 without satisfying (25), since the right side of (25) tends to 0 as $k \to \infty$, by (16).

This completes the proof.

IV. PROOF OF THEOREM C.

4.1. Let us now assume that $C\{M_n\}$ is non-quasianalytic and inverse-closed. By Theorem B, $\{A_n\}$ is then almost increasing, and so is $\{\alpha_n\}$, if $\alpha_n = M_n^{1/n}/n$. Choose K so that $\alpha_s \le K\alpha_n$ if $s \le n$.

Since $\sum M_n^{-1/n} < \infty$ (see § 1.3), $\sum (n\alpha_n)^{-1} < \infty$. But

$$\sum_{n^{1/2} \le s \le n} \frac{1}{s\alpha_s} \ge \frac{1}{K\alpha_n} \cdot \sum \frac{1}{s} \sim \frac{1}{K\alpha_n} \cdot \frac{1}{2} \log n .$$

The sum on the left tends to 0 as $n \to \infty$, hence $\alpha_n / \log n \to \infty$, and this means that $C\{M_n\}$ contains $C\{(n \log n)^n\}$ and therefore proves one half of Theorem C.

4.2. To prove the other half, we consider a function $f \notin C\{n \log n)^n\}$, and we shall construct a non-quasianalytic class $C\{M_n\}$, with $\{\alpha_n\}$ increasing, such that $f \notin C\{M_n\}$.

Since $f \notin C\{(n \log n)^n\}$, either some derivative of f fails to be bounded, in which case f belongs to no $C\{M_n\}$, or there is a sequence $\{n_i\}$ such that

(1) $$\|D^{n_i} f\| > (i^3 n_i \log n_i)^{n_i} ;$$

we can make $\{n_i\}$ increase so rapidly that

(2) $$n_{i+1} > n_i \log (i^2 \log n_i) .$$

Define

(3) $$\phi(n_i) = n_i \log (i^2 n_i \log n_i)$$

and

(4) $$\phi(n) = a_i + b_i n + n \log n \qquad (n_i \le n \le n_{i+1}) ,$$

where a_i and b_i are so chosen that the definitions of $\phi(n)$ agree when

$n = n_i$, $n = n_{i+1}$. Thus

(5) $$a_i + b_i n_i = n_i \log (i^2 \log n_i)$$
$$a_i + b_i n_{i+1} = n_{i+1} \log ((i+1)^2 \log n_{i+1}) .$$

From this we deduce that $a_i < 0$, and, via (2), that

(6) $$b_i > \log (i^2 \log n_{i+1}) - 1 .$$

Now put $M_n = \exp\{\phi(n)\}$. If $n_i \le n \le n_{i+1}$, then

(7) $$\exp\{-b_i\} < e/i^2 \log n_{i+1} ,$$

and hence, by (6),

(8) $$\frac{M_n}{M_{n+1}} = \exp\{\phi(n) - \phi(n+1)\} = \exp\{-b_i\} \cdot \frac{n^n}{(n+1)^{n+1}} < \frac{e}{i^2 \log n_{i+1}} \cdot (1 + \frac{1}{n})^{-n-1} \cdot \frac{1}{n} < \frac{1}{n\, i^2 \log n_{i+1}} .$$

It follows that

(9) $$\sum_{n_i+1}^{n_{i+1}} \frac{M_{n-1}}{M_n} < \frac{1}{i^2 \log n_{i+1}} \sum_{n_i}^{n_{i+1}} \frac{1}{n} < \frac{1}{i^2} ,$$

so that $C\{M_n\}$ is non-quasianalytic.

Next,

(10) $$\alpha_n = \frac{\phi(n)}{n} - \log n = b_i + \frac{a_i}{n} \qquad (n_i \le n \le n_{i+1}) ,$$

and since $a_i < 0$, $\{\alpha_n\}$ increases. We can also arrange our construction so that $b_{i+1} > b_i$, and then ϕ will be convex. (This is not really necessary, since the convergence of $\sum M_n/M_{n+1}$ assures the non-quasianalyticity of $C\{M_n\}$ even without logarithmic convexity of $\{M_n\}$.)

By (1) and (3), $f \notin C\{M_n\}$, and the proof of Theorem C is thus complete.

4.3. THEOREM. <u>The intersection of all non-quasianalytic classes</u> $C\{M_n\}$ <u>is the class</u> $C\{n!\}$. (Our reason for including a proof of this result is stated in § 1.6.)

<u>Proof.</u> If $A_{n_i} < A$ for some sequence $\{n_i\}$ tending to ∞ and some constant A, if $f \in C\{M_n\}$, and if $D^n f(0) = 0$ for $n = 0, 1, 2, \ldots$, then for each $x \neq 0$ there exists $\xi = \xi(x, n_i)$ such that

$$|f(x)| = |D^{n_i} f(\xi) x^{n_i}/n_i!| \leq |\beta B^{n_i} M_{n_i} x^{n_i}/n_i!|$$

$$= |\beta| \cdot |BA_{n_i} x|^{n_i} \leq |\beta| \cdot |BAx|^{n_i} ,$$

where β, B depend on f. If $|BAx| < 1$, it follows that $f(x) = 0$. Hence $C\{M_n\}$ is quasianalytic.

Thus $C\{n!\}$ is contained in every non-quasianalytic $C\{M_n\}$.

To prove the converse, suppose $f \notin C\{n!\}$. Then there is a sequence $\{n_i\}$ such that

$$\|D^{n_i} f\| > (i^3 n_i)^{n_i}$$

and

$$n_{i+1} > n_i \log (i^2 n_i) .$$

Put $\phi(n_i) = n_i \log (i^2 n_i)$, $\phi(n) = a_i + b_i n$ for $n_i \leq n \leq n_{i+1}$, where

$$a_i + b_i n_i = n_i \log (i^2 n_i)$$

$$a_i + b_i\, n_{i+1} = n_{i+1} \log ((i + 1)^2 n_{i+1}) ,$$

and define $M_n = \exp\{\phi(n)\}$. As in § 4.2, we now have $b_i > \log(i^2 n_{i+1}) - 1$, hence

$$\frac{M_n}{M_{n+1}} = e^{-b_i} < \frac{e}{i^2 n_{i+1}} \qquad (n_i \le n \le n_{i+1}) ,$$

and

$$\sum_{n_i+1}^{n_{i+1}} M_{n-1}/M_n < i^{-2} .$$

Thus $C\{M_n\}$ is non-quasianalytic, and since our definition of ϕ shows that $f \notin C\{M_n\}$, the proof is complete.

V. MISCELLANEOUS RESULTS

5.1. THEOREM. Every non-quasianalytic algebra $C\{M_n\}$ is contained in an inverse-closed algebra $C\{M^*_n\}$ which is minimal in the following sense: if $C\{M'_n\}$ contains $C\{M_n\}$ and if $C\{M'_n\}$ is inverse-closed, then $C\{M'_n\}$ contains $C\{M^*_n\}$.

Proof. Put $A^*_n = \max_{s \le n} A_s$ and $M^*_n = n!\, A^*_n$. Since $M_n \le M^*_n$ we have $C\{M_n\} \subset C\{M^*_n\}$. Since $\{A^*_n\}$ increases, $C\{M^*_n\}$ is inverse-closed. (Note that the proof of Theorem A made no use of logarithmic convexity.)

Now suppose $C\{M_n\} \subset C\{M'_n\}$ and $C\{M'_n\}$ is inverse-closed. Since $C\{M'_n\}$ is non-quasianalytic, Theorem B shows that $\{A'_n\}$ is almost increasing, where $A'_n = \{M'_n/n!\}^{1/n}$. Hence there are constants λ, K, such that $M_n \le \lambda^n M'_n$ and $A'_s \le KA'_n$ if $s \le n$. This implies $A_s \le \lambda K A'_n$, hence $A^*_n \le \lambda K A'_n$, hence $M^*_n \le (\lambda K)^n M'_n$, and thus $C\{M^*_n\} \subset C\{M'_n\}$.

5.2. THEOREM. <u>There exist non-quasianalytic algebras</u> $C\{M_n\}$ <u>which contain no inverse-closed non-quasianalytic</u> $C\{M'_n\}$.

<u>Proof.</u> Theorem 4.3 shows that there is a non-quasianalytic $C\{M_n\}$ such that

$$\left\{\frac{M_{n_i}}{n_i \log n_i}\right\}^{1/n_i} \to 0$$

for some sequence $\{n_i\}$. If $C\{M'_n\} \subset C\{M_n\}$, it follows that $C\{M'_n\}$ does not contain $C\{(n \log n)^n\}$, and hence Theorem C shows that $C\{M'_n\}$ cannot be both inverse-closed and non-quasianalytic.

5.3. COMPLEX HOMOMORPHISMS OF $C_p\{M_n\}$.

Since we are investigating certain function algebras, it is appropriate to study their maximal ideals and the complex homomorphisms which exist on them. We restrict ourselves to the algebras $C_p\{M_n\}$, for simplicity, for then we are dealing with functions on the circle T, i.e., on a compact space.

If $C_p\{M_n\}$ is inverse-closed, there are no problems. For each $x \in T$, let I_x be the set of all $f \in C_p\{M_n\}$ which vanish at x. Then I_x is clearly a maximal ideal in $C_p\{M_n\}$. Conversely, assume I is a maximal ideal different from every I_x. For each x, there is a function $f_x \in I$ such that $f_x(x) \neq 0$, and the compactness of T shows that there are points $x_1, \ldots, x_n$ such that $g = \sum_1^n f_{x_i} \bar{f}_{x_i} > 0$. But $g \in I$, and since $C_p\{M_n\}$ is inverse-closed (by assumption), we have $1 \in I$, hence $I = C_p\{M_n\}$. We summarize:

<u>If</u> $C_p\{M_n\}$ <u>is inverse-closed, then every maximal ideal</u> I <u>in</u> $C_p\{M_n\}$ <u>is of the form</u> $I = I_x$, <u>and every complex homomorphism</u> ψ <u>of</u> $C_p\{M_n\}$ <u>is of the form</u> $\psi(f) = f(x)$, <u>for some</u> $x \in T$.

(By a complex homomorphism of $C_p\{M_n\}$ we mean a multiplicative linear functional which maps $C_p\{M_n\}$ onto the complex field. We make no continuity assumptions. Indeed, we have not introduced a topology in $C_p\{M_n\}$.)

If $C_p\{M_n\}$ is not inverse-closed, then, on the other hand, there do also exist other maximal ideals. For if $f \in C_p\{M_n\}$, if f has no zero on T, and if $1/f \notin C_p\{M_n\}$, then f generates a proper ideal in $C_p\{M_n\}$ which, by Zorn's lemma, is contained in a maximal ideal I; since $f \in I$, I is different from I_x for all $x \in T$.

It is nevertheless conceivable that all complex homomorphisms are of the form $\psi(f) = f(x)$ for some $x \in T$, so that the quotient algebras $C_p\{M_n\}/I$ are different from the complex field, whenever I is not one of the ideals I_x.

We shall now prove that this conjecture is true, under the additional assumption that $C_p\{M_n\}$ is non-quasianalytic and that $\log M_n = 0(n^2)$. We divide the proof into several steps. Our growth condition will only be used at the end.

We consider a fixed $C_p\{M_n\}$, and a fixed complex homomorphism ψ of $C_p\{M_n\}$.

(i) There is a point $x_0 \in T$ such that $\psi(f) = 0$ for all $f \in C_p\{M_n\}$ which vanish near x_0, (i.e., in a neighborhood of x_0).

For if there is no such point, the compactness of T shows that there are segments $V_1, \ldots, V_m$ and functions $f_1, \ldots, f_m$ such that $f_i = 0$ on V_i but $\psi(f_i) = 1$. Putting $f = f_1 \ldots f_m$, we have $f = 0$, $\psi(f) = \psi(f_1) \ldots \psi(f_m) = 1$, and hence $\psi(0) = 1$, a contradiction.

For simplicity, we assume from now on that $x_0 = 0$.

(ii) Suppose $f \in C_p\{M_n\}$ and $f(x) = x$ near 0. Then $\psi(f) = 0$.

Proof. Put $\psi(f) = \alpha$. If $\alpha \neq 0$, then there exists $g \in C_p\{M_n\}$ such that $g(x) = (x - \alpha)^{-1}$ near 0; this is so since $(x - \alpha)^{-1}$ is analytic near 0, and we can multiply by one of functions h constructed in Lemma 3.2.

Then $(f - \alpha)\cdot g = 1$ near 0, and (i) shows that $\psi(f - \alpha)\psi(g) = 1$. But $\psi(f - \alpha) = \psi(f) - \alpha = 0$, a contradiction.

(iii) If $f \in C\{M_n\}$, $f(0) = 0$, and $g(x) = f(x)/x$, then $g \in C\{M_{n+1}\}$.

Proof. Repeated differentiation of the equation $f(x) = xg(x)$ yields

$$D^{n+1}f(x) = xD^{n+1}g(x) + (n+1)D^n g(x) \qquad (n \geq 0) .$$

As $|x| \to \infty$, $D^n g(x) \to 0$, and $D^{n+1}g(x) = 0$ at every local maximum of $|D^n g|$. Hence $\|D^n g\| \leq \|D^{n+1}f\|$.

(iv) If $f \in C_p\{M_n\}$ and $f(0) = 0$, then $\psi(f) = 0$.

Proof. There are functions $g, h \in C_p\{M_n\}$ such that $g \equiv 1$ near 0, the support of g lies in $[-\pi + \delta, \pi - \delta]$ for some $\delta > 0$, and $h(x) = x$ on the support of g.

Put $F = fg/h$. Since $h = x$ where $fg \neq 0$, $F = fg/x$. Since $fg \in C_p\{M_n\}$, (iii) shows that $F \in C_p\{M_{n+1}\}$. But if $\log M_n = 0(n^2)$, then $C\{M_{n+1}\} = C\{M_n\}$ [1; p. 22]. Thus $F \in C_p\{M_n\}$.

By (i), $\psi(g) = 1$; by (ii), $\psi(h) = 0$. Hence $\psi(f) = \psi(f)\psi(g) = \psi(fg) = \psi(Fh)$ $= \psi(F)\psi(h) = 0$.

We now summarize the result:

THEOREM. If $C_p\{M_n\}$ is non-quasianalytic, if $\log M_n = 0(n^2)$, and if ψ is a complex homomorphism of $C_p\{M_n\}$, then $\psi(f) = f(x)$ for some $x \in T$.

We conclude with the remark that there exist non-quasianalytic algebras $C\{M_n\}$ which are not inverse-closed and which fail to satisfy the condition $\log M_n = 0(n^2)$. (In fact, if $\omega_n \to \infty$ and if $\lambda_n/n! \to \infty$, the technique used in the proof of Theorem 4.3 allows us to construct non-quasianalytic $C\{M_n\}$ such that $M_n > \omega_n$ for infinitely many n, and also $M_n < \lambda_n$ for infinitely many n.) For these algebras we do not yet know all complex homomorphisms.

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